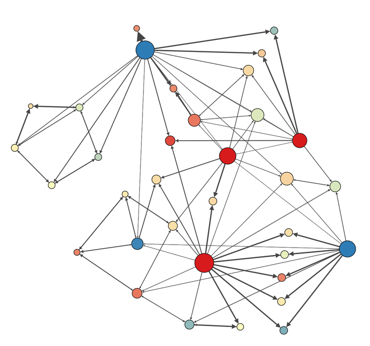
Polar Opinion Dynamics in Social Networks

Victor Amelkin, Francesco Bullo, and Ambuj K. Singh
University of California, Santa Barbara, CA, United States of America

Introduction



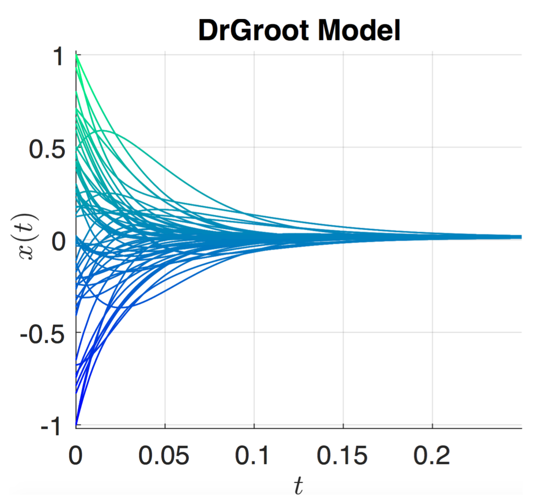
- Social network consisting of *agents*
- Each agent holds an opinion
- Opinions change due to the agents' interaction
- Opinions are *polar* (e.g., 🐴 vs. 🐘)
- Objective—modeling polar opinion formation

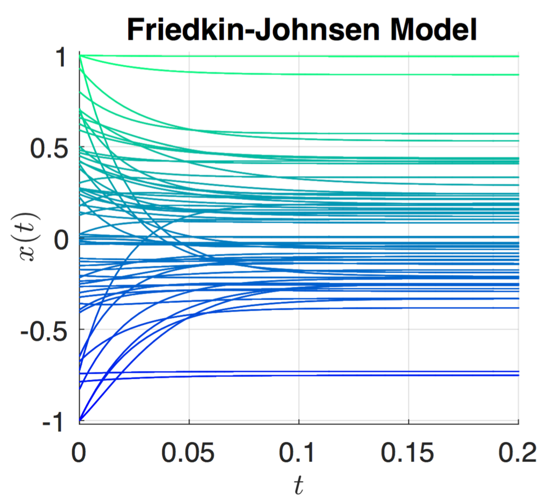
Problem Statement

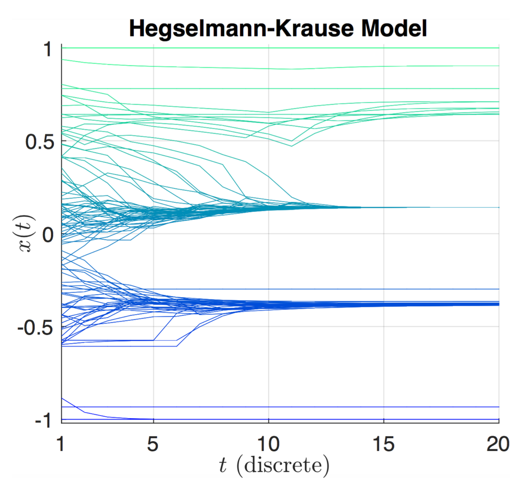
- Directed, strongly connected social network of n agents
- Row-stochastic adjacency matrix $\boldsymbol{W} \in [0,1]^{n \times n}$
- $(\boldsymbol{W})_{ij} = w_{ij}$ – how much agent i relatively "trusts" agent j
- $\boldsymbol{x}(t) \in [-1,1]^n$ – agents' opinions at time t
- Goals:
  - design a sociologically plausible model governing evolution of $x(t)$
    $$\boldsymbol{x}(t+1) = M(\boldsymbol{x}(t), t) \quad \text{or} \quad d\boldsymbol{x}/dt = \dot{\boldsymbol{x}}(t) = M(\boldsymbol{x}(t), t)$$
    that necessarily captures the *competing nature of opinions*;
  - and analyze the dynamical behavior of the model to understand the dependency of $\boldsymbol{x}(\infty)$ upon $\boldsymbol{x}(0)$ and the network's structure.

Background – Models for Homogeneous Opinions

DeGroot	$\boldsymbol{x}(t+1) = \boldsymbol{W}\boldsymbol{x}(t)$ $(\dot{x} = -Lx = \Delta x)$
Time-varying DeGroot	$\boldsymbol{x}(t+1) = \boldsymbol{W}(t)\boldsymbol{x}(t)$
Friedkin-Johnsen	$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{W}\boldsymbol{x}(t) + (\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}(0)$
	$\boldsymbol{A}$ – diagonal matrix of susceptibilities
Bounded Confidence	$\boldsymbol{x}(t+1) = \boldsymbol{W}(\boldsymbol{x}(t))\boldsymbol{x}(t)$
	$w_{ij}(x) > 0 \Leftrightarrow \lvert x_i - x_j \rvert \le \textit{bound}$







Models for Polar Opinion Dynamics

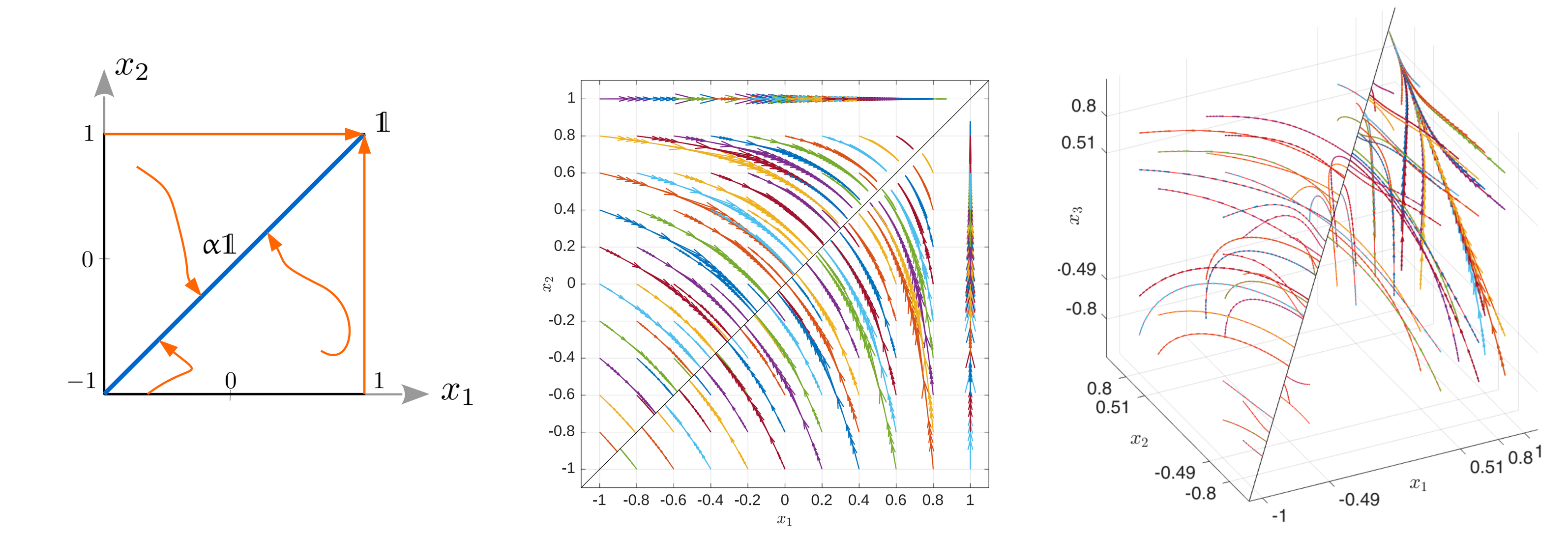
- *Key idea* – agents' opinion formation behavior must change based on the agents' current beliefs
  $$\boldsymbol{x}(t+1) = \boldsymbol{A}(\boldsymbol{x}(t))\boldsymbol{W}\boldsymbol{x}(t) + (\boldsymbol{I} - \boldsymbol{A}(\boldsymbol{x}(t)))\boldsymbol{x}(t)$$
- **General model for polar opinion dynamics:**
  $$\dot{\boldsymbol{x}} = -\boldsymbol{A}(\boldsymbol{x}(t))\boldsymbol{L}\boldsymbol{x}$$
  where
  $$\boldsymbol{x} = \boldsymbol{x}(t) \in [-1,1]^n, \ \ \boldsymbol{A}(\boldsymbol{x}(t)) \in \mathbf{diag}([0,1]^n),$$
  $\boldsymbol{L} = (\boldsymbol{I} - \boldsymbol{W})$ – out-degree Laplacian of the network
- **Model with stubborn positives**
  $$\dot{\boldsymbol{x}} = -1/2\,(\boldsymbol{I} - \mathbf{diag}(\boldsymbol{x}))\boldsymbol{L}\boldsymbol{x}$$
- **Model with stubborn neutrals**
  $$\dot{\boldsymbol{x}} = -\,\mathbf{diag}(\boldsymbol{x})^2\boldsymbol{L}\boldsymbol{x}$$
- **Model with stubborn extremists**
  $$\dot{\boldsymbol{x}} = -(\boldsymbol{I} - \mathbf{diag}(\boldsymbol{x})^2)\boldsymbol{L}\boldsymbol{x}$$

Select Theoretical Results

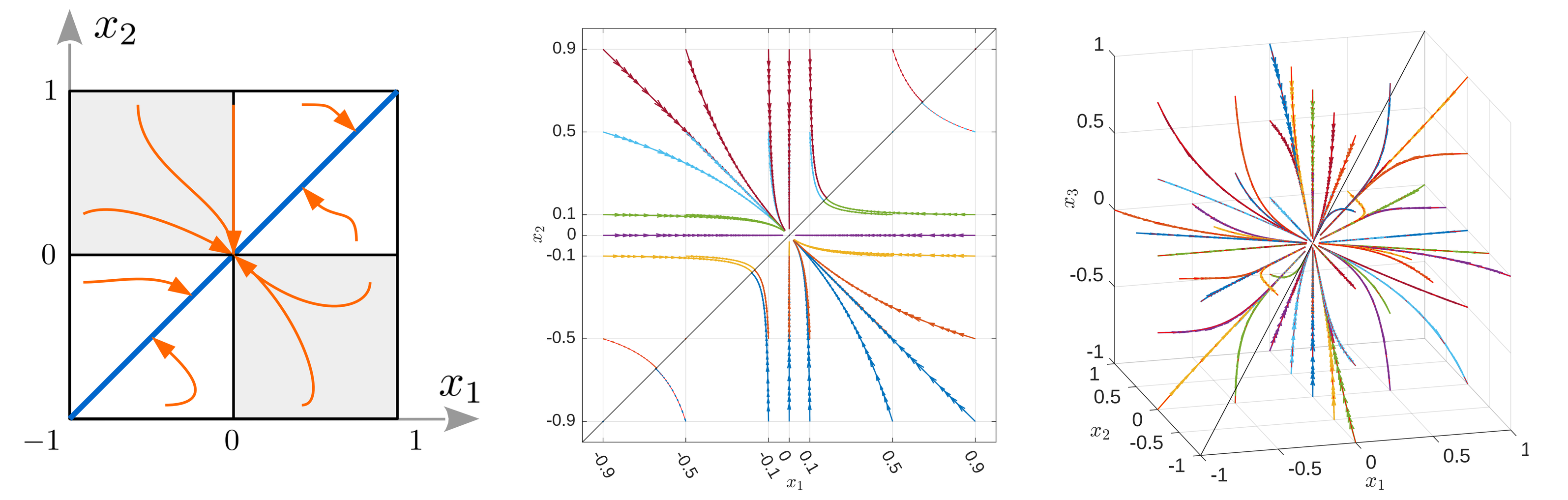
Theorem (General convergence to consensus):
Suppose W is a row-stochastic adjacency matrix of a strongly connected network $G(W)$, and agent states $x(t) \in [-1,1]^n$ evolve according to the general model of polar opinion dynamics $\dot{x} = f(x) = -A(x)Lx$, where the agents potentially have different susceptibility functions $A_{ii}(x)$. Let $S \subseteq [-1,1]^n$ be a non-empty compact set, forward invariant w.r.t. the model, and $N = S \cap \{\alpha\mathbb{1} \mid \alpha \in [-1,1]\}$ be its non-empty subset of consensus states. Further, assume that in S, the agents' susceptibility functions $A_{ii}(x)$ agree upon their zeros in that
$$\forall x \in S\ \forall i,j \in \{1,\ldots,n\} : A_{ii}(x) = A_{jj}(x) = 0 \rightarrow x_i = x_j.$$
Then, all trajectories $x(t)$ of the model starting in S converge to N as $t \rightarrow \infty$.

Models' Behavior

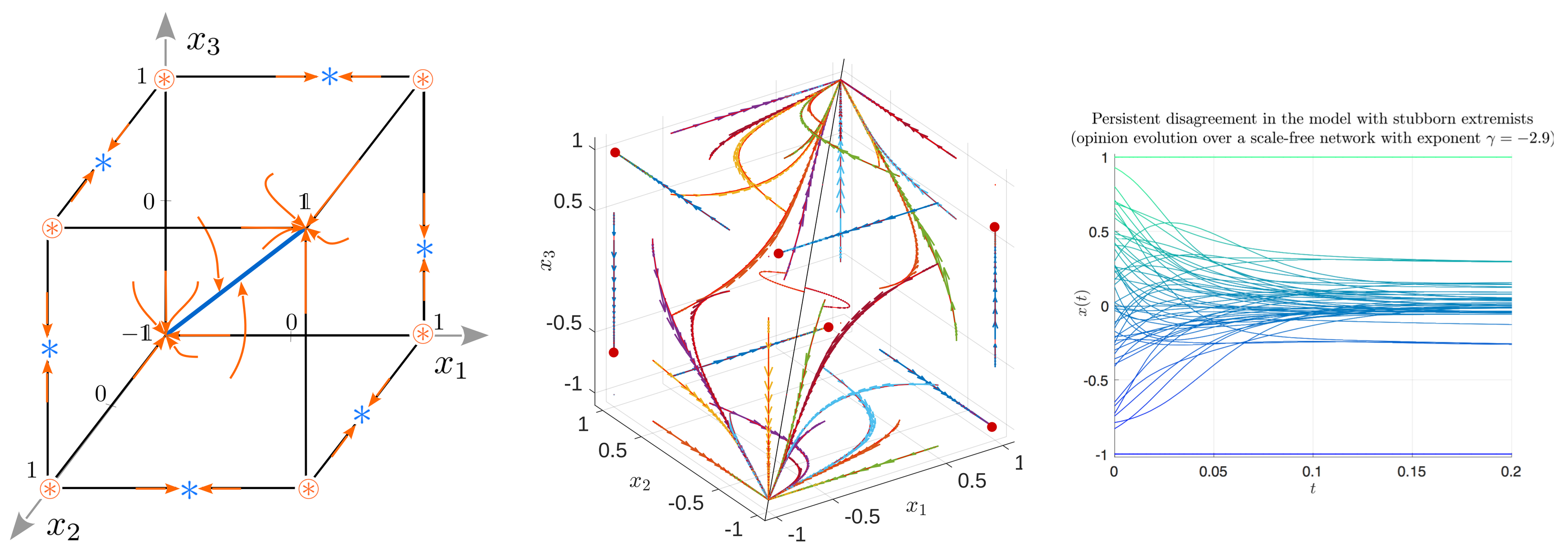
Model with stubborn positives:



Model with stubborn neutrals:



Model with stubborn extremists:



Summary

- Proposed non-linear models—the first theoretically analyzed models suitable for the case of polar opinions.
- Agents approach consensus in the absence of competing groups of ultimately stubborn agents.
- In the model with stubborn extremists, ultimately stubborn agents holding different opinions "drive" other agents to disagreement
  $$\boldsymbol{x}(\infty) = \boldsymbol{P}^{\mathsf{T}}[\boldsymbol{x}_1(0)^{\mathsf{T}}, (\boldsymbol{I} - \boldsymbol{W}_{22})^{-1}\boldsymbol{W}_{21}\boldsymbol{x}_1(0)^{\mathsf{T}}]^{\mathsf{T}},$$
  where index 1 corresponds to the ultimately stubborn agent group.

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{victor,ambuj}@cs.ucsb.edu, bullo@engineering.ucsb.edu