CS40-Summer 2015 Notes

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Combinatorics

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① **The rule of product:**

(order matters)

1) $A = \{1, 2, 3\}$ — 2-cell array

$\# = 3 \cdot 3 = 9$

2) $A = \{1, 2\}$, $B = \{1, 2, 3\}$

$\# = 2 \cdot 3 \cdot 2 = 12$

3) $\Sigma = \{0, 1\}$ — binary string of length n, $n \in \mathbb{N}$

$\#_n = 2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$

$S = \{a_1, a_2, \ldots, a_n\}$

$|P(S)| = 2^n$ (# of subsets of set S)

Subset $\sim$

a_1	a_2	a_3	$\ldots$	a_{n-1}	a_n
1	0	1		1	0

$a_1 \in$ subset, $a_2 \notin$ subset

4) # of palyndromes of length n over $\Sigma = \{a, b, c\}$

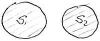
$\underbrace{\boxed{a\,|\,b\,|\,b\,|\,c}}_{n/2}\boxed{c\,|\,b\,|\,b\,|\,a}$, $\underbrace{\boxed{a\,|\,c\,|\,b}}_{\lceil n/2 \rceil}\boxed{c\,|\,a}$

$\# = |\Sigma|^{\lceil n/2 \rceil} = 3^{\lceil n/2 \rceil}$

② **The rule of sum:**

def: $S_1, S_2, \ldots, S_n$ — n disjoint sets ($S_i \cap S_j = \emptyset$, $i \neq j$)

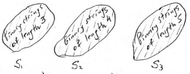
$|S_1 \cup S_2 \cup \ldots \cup S_n| = \sum_{i=1}^{n} |S_i|$



$|S_1 \cup S_2| = |S_1| + |S_2|$

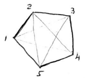
1) # of binary strings of length $3 \le n \le 5$:

$\# = \#_3 + \#_4 + \#_5$



$\# = |S_1 \cup S_2 \cup S_3| = \sum |S_i| = 2^3 + 2^4 + 2^5$

2) **Does not work:**



n-sided convex polygon

# of diagonals – ?

S_i – diagonals starting at point i

Unfortunately, $S_i \cap S_j \neq \emptyset$, $i \neq j$ (for some i, j), but still can count:

$\# = \frac{\#_1 + \ldots + \#_n}{2}$, $\#_i = (n-3)n/2$

See a separate document for the problem with coverings with 2-dominoes and Fibonacci numbers

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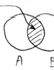
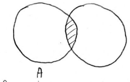
③ Inclusion - exclusion - "the sum rule for overlapping sets"

$A \cap B \neq \emptyset$



$$|A \cup B| = |A| + |B| - |A \cap B|$$

1) # of binary strings of length n starting with 1 or ending with 0:



B - binary strings of len n ending with 0

Ex: $\underbrace{\boxed{1}\boxed{0}\boxed{1}\boxed{\cdots}\boxed{0}\boxed{0}}_{n}$

A - binary strings of len n starting with 1

Ex: $\underbrace{\boxed{1}\boxed{0}\boxed{1}\boxed{1}\boxed{\cdots}\boxed{0}}_{n}$

$(A \cap B)$ - binary strings of length n both starting with 1 and ending with 0

Ex: $\underbrace{\boxed{1}\boxed{0}\boxed{1}\boxed{1}\boxed{0}\boxed{\cdots}\boxed{0}}_{n}$

$A \cap B \neq \emptyset \Rightarrow$ Cannot use sum rule; need inclusion - exclusion:

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A| = |\{\boxed{1}\underbrace{\boxed{?}\boxed{?}\boxed{\cdots}\boxed{?}\boxed{?}}_{n-1}\}| = 2^{n-1}$

$|B| = |\{\underbrace{\boxed{?}\boxed{?}\boxed{\cdots}\boxed{?}}_{n-1}\boxed{0}\}| = 2^{n-1}$

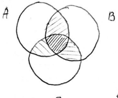
$|A \cap B| = |\{\boxed{1}\underbrace{\boxed{?}\boxed{?}\boxed{\cdots}\boxed{?}}_{n-2}\boxed{0}\}| = 2^{n-2}$

$\# = |A \cup B| = 2^{n-1} + 2^{n-1} - 2^{n-2} = \underbrace{2^n - 2^{n-2}}_{} = 4 \cdot 2^{n-2} - 2^{n-2} = 3 \cdot 2^{n-2}, \quad n \geq 2$

$\# = 0$ if $n < 2$

Get the same result by complementation! (#good = #all - #bad)

2) General statement for 3 sets A, B, C:



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

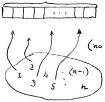
Ex: $x \in A$ ~ "x is a password having at least 1 digit"

④ Permutations:

$[n] = \{1, 2, 3, \ldots, n\}$ - elements

$3\ 5\ 1\ 7\ (n-1)\ \ldots\ 9\ n$ - example of a permutation of $[n]$ (all elements present)

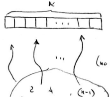
\# of permutations of $[n]$ - ?



$\# = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 = n!$

(no repetitions!)

\# of ordered arrangements of size k out of $[n]$:



$$\# = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = \frac{n\,(n-1)\,(n-2)\ldots(n-k+1)\,\cdot\,(n-k)\,(n-k-1)\ldots 2 \cdot 1}{(n-k)\,(n-k-1)\ldots 2 \cdot 1}$$

(no repetitions)

$$= \frac{n!}{(n-k)!} = P(n,k)$$

⑤ Combinations: ("Like $P(n,k)$, but order does not matter now: $1\,2\,3 = 2\,1\,3$")

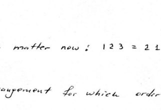
Combination of k elements out of $[n]$ is an arrangement for which order does not matter.

$C(n,k)$ - \# of combinations

$$C(n,k) = \underbrace{P(n,k)}_{\text{count all ordered arrangements}} \Big/ \underbrace{k!}_{\text{for every arrangement, remove those that differ only by the order of elements}} = \frac{n!}{k!\,(n-k)!}$$

$C(n,k) = C_n^k = \boxed{\binom{n}{k}}$ - alternative notations

Binomial coefficient



$C_3^2 = \binom{3}{2} = 3$

$\binom{4}{2} = 6$

Pascal Triangle

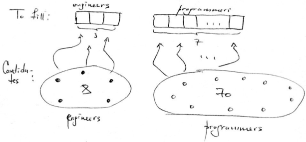
Symmetry: $\binom{n}{k} = \binom{n}{n-k}$

Pascal's identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

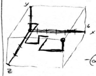
1) Company wants to hire 3 engineers and 7 programmers. There are 8 engineers and 70 programmers among the candidates. # of hiring decisions (from the company's point of view)?

To fill:



$$\# = \binom{8}{3} \times \binom{70}{7}$$

A better (though, similar) example for multinomial coefficients:



$x \in [0;6]$
$y \in [0;3]$
$z \in [0;4]$
- (discrete) 3d-grid

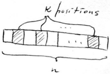
# of paths from $\langle 0,0,0 \rangle$ to $\langle 6,3,4 \rangle$, when on each step, coordinates cannot decrease (monotonic "non-decreasing" paths)

$$\# = \frac{(6+3+4)!}{6!\,3!\,4!}$$

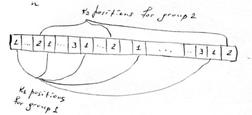
$$\binom{n}{k_1\ k_2\ \dots\ k_m} = \frac{n!}{k_1!\,k_2!\,\dots\,k_m!}$$ - multinomial coefficient ("Multinomial theorem"? :))

$\sum k_i = n$

$\binom{n}{k}$ - choosing k positions in an n-long array of cells

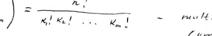


$\binom{n}{k_1\ k_2\ \dots\ k_m}$ - choosing m groups of k_i positions in group i



+

Ex: 2) There are 10 apples, 15 bananas, and 7 coconuts in my fridge. (Objects of the same kind, i.e., 2 apples, are identical.)
In how many ways I can arrange 5 apples, 2 bananas, and 3 coconuts in a line? (Order matters: $\langle$apple, banana, coconut$\rangle \neq \langle$ban., ap., coc.$\rangle$)



3 groups: $k_1 = 5$, $k_2 = 2$, $k_3 = 3$

$$\# = \binom{10}{5\ 2\ 3} = \frac{10!}{5!\,3!\,2!} = \frac{\cancel{6} \cdot 7 \cdot \cancel{8}^{4} \cdot 9 \cdot 10}{\cancel{2} \cdot \cancel{3} \cdot \cancel{2}} = 4 \cdot 7 \cdot 9 \cdot 10 = \text{?} :)$$

Notice: it does not matter how many objects are in the fridge (as long as there are enough!)

3) $S = \{a_1, a_2, \ldots, a_n\}$

$|P(S)| = \#$ of subsets of S = ?

$$\# = \underbrace{\binom{n}{0}}_{\substack{\#\text{ of subsets}\\ \text{of size } 0}} + \underbrace{\binom{n}{1}}_{\substack{\#\text{ of subsets}\\ \text{of size } 1}} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} =$$

$$= (1+1)^n = 2^n$$

$\left(\begin{array}{l} \text{recall } (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}; \\ \text{try } x = y = 1 \end{array}\right)$

Combinations with repetitions are covered in a separate document.

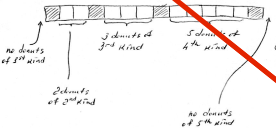
⑥ ~~Combinations with repetitions~~ : ~~(a bad name for partitioning a number into a certain number of (possibly empty) parts)~~

~~* order does not matter; matter only numbers of identical objects of certain kind~~

~~* "repetitions allowed" (~ "infinite supply of objects of each kind")~~

~~1) 5 kinds of doughnuts (infinite supply; donuts of the same kind are identical)~~

~~\# of ways to select 10 donuts - ?~~



10 cells for doughnuts
+
4 cells for kind separators
(5-1)

$\#$ = # of ways to fit 4 ▨-cells in 14 □-slots = $\binom{14}{4} = \binom{14}{10}$

Note: if parts are not allowed to be empty, then the result will be completely different (and expressed in terms of Stirling numbers of second kind)!

In general, $\# = \binom{n + r - 1}{r}$ (e.g., $\binom{5 + 10 - 1}{10}$)

n = # of types of objects

r = # of objects to select

2) In how many ways integer 10 can be partitioned into 5 possibly empty parts? (Ex: $\underset{\text{part 1}}{1} + 2 + \underset{\text{part 3}}{0} + \underset{\text{part 4}}{7} + 0 = 10$)

$$\# = \binom{10+5-1}{10}$$