Polar Opinion Dynamics in Social Networks

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Motivation

- Social network consisting of *agents*
- Each agent holds an opinion
- Agents' opinions change due to the agents' interaction
- Opinions are *polar* (*iOS vs. Android*, *Republicans vs. Democrats*)
- Goal: model opinion formation and learn from the model

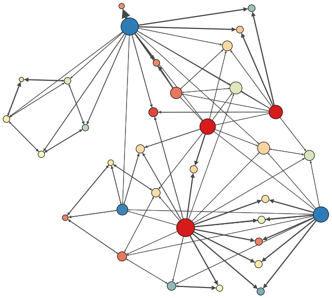


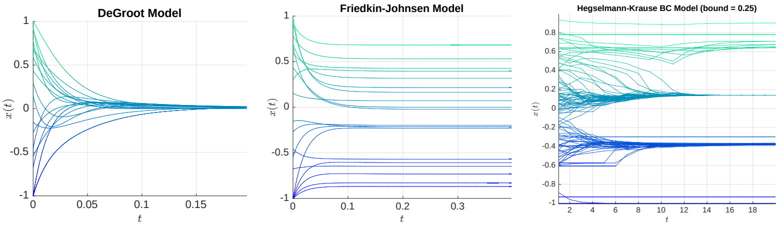
Figure: *Zachary's Karate Club* network [Zac77]

Problem statement

- Directed, strongly connected social network of n agents
- Row-stochastic adjacency matrix $\boldsymbol{W} \in [0,1]^{n\times n}$
- $(\boldsymbol{W})_{ij} = w_{ij}$ – how much agent i relatively "trusts" agent j
- $\boldsymbol{x}(t) \in [-1,1]^n$ – agents' opinions at time t
- Goals:
  - Design a sociologically plausible model governing evolution of $x(t)$
    $$\dot{\boldsymbol{x}}(t) = M(\boldsymbol{x}(t), t)$$
  - The model *must* capture the competing nature of opinions.
  - Analyze the dynamical behavior of the model to understand the dependency of $\boldsymbol{x}(\infty)$ upon $\boldsymbol{x}(0)$ and $\boldsymbol{W}$.

Existing opinion dynamics models

DeGroot [DeG74]	$\boldsymbol{x}(t+1) = \boldsymbol{W}\boldsymbol{x}(t) \quad (\dot{x} = -\boldsymbol{L}x,\ \dot{x} = \Delta x)$
Time-varying DeGroot [Mor05]	$\boldsymbol{x}(t+1) = \boldsymbol{W}(t)\boldsymbol{x}(t)$
Friedkin-Johnsen [FJ99]	$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{W}\boldsymbol{x}(t) + (\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}(0)$ $\boldsymbol{A}$ – diagonal matrix of susceptibilities
Bounded Confidence [HK02, Lor07]	$\boldsymbol{x}(t+1) = \boldsymbol{W}(\boldsymbol{x}(t))\boldsymbol{x}(t)$ $w_{ij}(x) > 0 \Leftrightarrow \lvert x_i - x_j \rvert \leq bound$



Other models: SI/SIS/SIR-like, Independent Cascade, Linear Threshold, Voter, Bayesian, . . .

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General model for polar opinion dynamics

- *Key idea* – agents' opinion formation behavior must change based on the agents' current beliefs

$$\boldsymbol{x}(t+1) = \boldsymbol{A}(\boldsymbol{x}(t))\boldsymbol{W}\boldsymbol{x}(t) + (\boldsymbol{I} - \boldsymbol{A}(\boldsymbol{x}(t)))\boldsymbol{x}(t)$$

General model for polar opinion dynamics

- *Key idea* – agents' opinion formation behavior must change based on the agents' current beliefs

$$
\begin{aligned}
\boldsymbol{x}(t+1) &= \boldsymbol{A}(\boldsymbol{x}(t))\boldsymbol{W}\boldsymbol{x}(t) + (\boldsymbol{I} - \boldsymbol{A}(\boldsymbol{x}(t)))\boldsymbol{x}(t) \\
\boldsymbol{x}(t+1) &= \boldsymbol{A}\boldsymbol{W}\boldsymbol{x}(t) + (\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}(0) \qquad \text{(FJ)}
\end{aligned}
$$

General model for polar opinion dynamics

- *Key idea* – agents' opinion formation behavior must change based on the agents' current beliefs

$$\boldsymbol{x}(t+1) = \boldsymbol{A}(\boldsymbol{x}(t))\boldsymbol{W}\boldsymbol{x}(t) + (\boldsymbol{I} - \boldsymbol{A}(\boldsymbol{x}(t)))\boldsymbol{x}(t)$$

- **General model for polar opinion dynamics [ABS17]:**

$$\boxed{\dot{\boldsymbol{x}} = -\boldsymbol{A}(\boldsymbol{x})\boldsymbol{L}\boldsymbol{x}}$$

where

$$\boldsymbol{x} = \boldsymbol{x}(t) \in [-1,1]^n,$$

$$\boldsymbol{A}(\boldsymbol{x}) : [-1,1]^n \to \mathbf{diag}([0,1]^n),$$

$$\boldsymbol{L} = (\boldsymbol{D}^{\,out} - \boldsymbol{W}) \text{ – out-degree Laplacian of the network.}$$

[ABS17] Amelkin V., Bullo F., Singh A.K. *"Polar Opinion Dynamics in Social Networks"*, IEEE Transactions on Automatic Control (2017)

Three instances of the general model

- **Model with stubborn positives:**

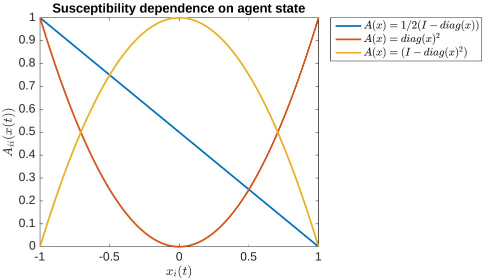
$$\boldsymbol{A}(\boldsymbol{x}) = 1/2\,(\boldsymbol{I} - \mathbf{diag}(\boldsymbol{x})), \quad \dot{\boldsymbol{x}} = -1/2\,(\boldsymbol{I} - \mathbf{diag}(\boldsymbol{x}))\boldsymbol{L}\boldsymbol{x}$$

- **Model with stubborn neutrals:**

$$\boldsymbol{A}(\boldsymbol{x}) = \mathbf{diag}(\boldsymbol{x})^2, \quad \dot{\boldsymbol{x}} = -\,\mathbf{diag}(\boldsymbol{x})^2\boldsymbol{L}\boldsymbol{x}$$

- **Model with stubborn extremists:**

$$\boldsymbol{A}(\boldsymbol{x}) = (\boldsymbol{I} - \mathbf{diag}(\boldsymbol{x})^2), \quad \dot{\boldsymbol{x}} = -(\boldsymbol{I} - \mathbf{diag}(\boldsymbol{x})^2)\boldsymbol{L}\boldsymbol{x}$$



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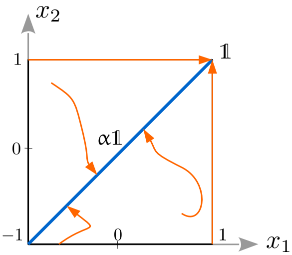
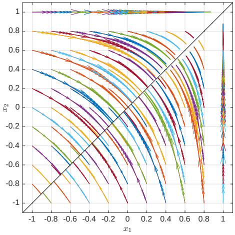
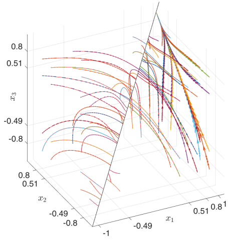
Convergence: model with stubborn positives $\dot{\boldsymbol{x}} = -1/2(\boldsymbol{I} - \mathbf{diag}(\boldsymbol{x}))\boldsymbol{L}\boldsymbol{x}$

- If $\boldsymbol{x}(0) < \mathbb{1}$, then $\lim_{t\to\infty} \boldsymbol{x}(t) = \alpha\mathbb{1}$ for some $\alpha \in [0, 1)$.
- If $\exists i \in \{1, \ldots, n\} : \boldsymbol{x}_i = 1$, then $\lim_{t\to\infty} \boldsymbol{x}(t) = \mathbb{1}$.





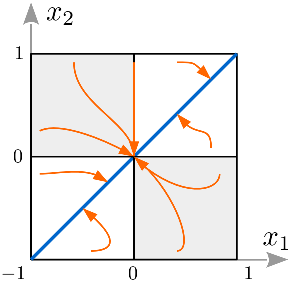


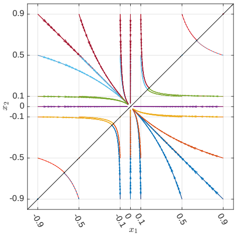
Convergence: model with stubborn neutrals $\dot{\boldsymbol{x}} = -\,\mathbf{diag}(\boldsymbol{x})^2 \boldsymbol{L}\boldsymbol{x}$

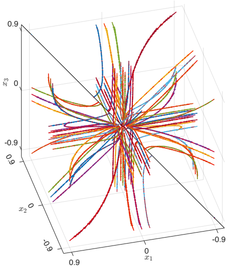
- If
  - $\boldsymbol{x} < \mathbb{0}$ (negative semi-open orthant) or
  - $\boldsymbol{x} > \mathbb{0}$ (positive semi-open orthant),

  then $\lim\limits_{t\to\infty} \boldsymbol{x}(t) = \alpha\mathbb{1}$ for some
  - $\alpha \in [-1, 0)$ or
  - $\alpha \in (0, 1]$, respectively.

- Otherwise, $\lim\limits_{t\to\infty} \boldsymbol{x}(t) = \mathbb{0}$.







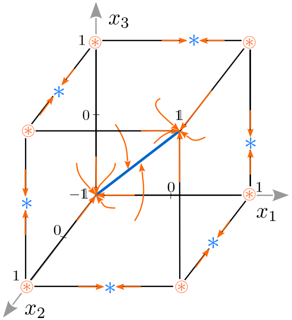
Convergence: model with stubborn extremists $\dot{\boldsymbol{x}} = -(\boldsymbol{I} - \mathbf{diag}(\boldsymbol{x})^2)\boldsymbol{L}\boldsymbol{x}$

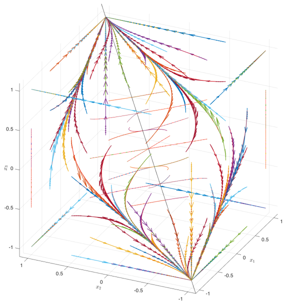
- If $|\boldsymbol{x}(0)| < \mathbb{1}$, then $\boldsymbol{x}(\infty) = \alpha\mathbb{1}$ for some $\alpha \in (-1, 1)$.

- If all closed agents in $\boldsymbol{x}(0)$ have the same state, -1 or 1, then $\boldsymbol{x}(\infty)$ is either $-\mathbb{1}$ or $\mathbb{1}$, respectively.

- If there are some closed agents having different opinions $\boldsymbol{x}_1(0)$, and there may be some open agents $\boldsymbol{x}_2(0)$, then

$$\boldsymbol{x}(\infty) = \boldsymbol{P}^{\mathsf{T}}[\boldsymbol{x}_1(0)^{\mathsf{T}}, (\boldsymbol{I} - \boldsymbol{W}_{22})^{-1}\boldsymbol{W}_{21}\boldsymbol{x}_1(0)^{\mathsf{T}}]^{\mathsf{T}}$$





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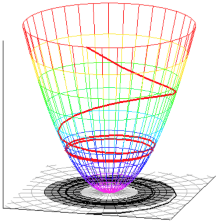
Lyapunov Direct Method [Lya92]

Find $V(\boldsymbol{x}) : D \to \mathbb{R} \in C^1(D)$ with a unique minimum at $\boldsymbol{x}^*$, whose directional derivative $\mathcal{L}_f(V(\boldsymbol{x})) = \langle \nabla V(\boldsymbol{x}), f(\boldsymbol{x}) \rangle$ along the trajectories of the system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ is non-positive and zero at $\boldsymbol{x}^*$. Then, $\boldsymbol{x}^*$ is asymptotically stable.

LaSalle Invariance Principle [LL61]

Given a compact $S \subseteq D$ forward-invariant w.r.t. system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, find $V(\boldsymbol{x}) : D \to \mathbb{R} \in C^1(D)$ with $\mathcal{L}_f(V(\boldsymbol{x})) \leq 0$. Then, trajectories starting in S converge to the largest invariant subset of the 0-level set of $\mathcal{L}_f(V(\boldsymbol{x}))$.

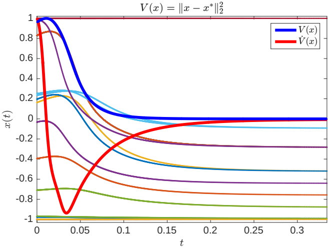


Quadratic Lyapunov function

(The image is due to Matt Kawski @ ASU)

When smooth analysis does not work

- Quadratic Lyapunov functions generally[1] do not work well for directed networks.
  - Formally, for a negative Laplacian flow $\dot{\boldsymbol{x}} = -\boldsymbol{L}\boldsymbol{x} = f(\boldsymbol{x})$, if $V(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{x}^*\|_2^2$, then $\mathcal{L}_f V(\boldsymbol{x}) \leq 0 \Leftrightarrow G(L)$ is weight-balanced.
  - Informally, while such $V(\boldsymbol{x})$ are often non-negative and 0 only at $\boldsymbol{x}^*$, their Lie derivative occasionally happens to be positive.



- Thus, it may worth time to look for a non-smooth Lyapunov function, even for a system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ with a continuous vector field f.

[1] Quadratic Lyapunov functions *may* work for directed networks; for example, paper [KBG14] shows and ad hoc design of one such function for a specific non-linear system – individual-level network SIS.

Let $V : \mathbb{R}^d \to \mathbb{R}$ be locally Lipschitz[2], and $\Omega_V \subset \mathbb{R}^d$ be the set of points where V fails to be differentiable.

Generalized Gradient [Cla90]:

$$\partial V(\boldsymbol{x}) = \mathbf{co}\left\{ \lim_{i\to\infty} \nabla V(\boldsymbol{x}_i) \mid \boldsymbol{x}_i \to \boldsymbol{x}, \boldsymbol{x}_i \notin N_0 \cup \Omega_V \right\},$$

$\mathbf{co}\{\ \}$ – convex hull,

N_0 – any set of Lebesgue measure zero.

In English:

- compute $\nabla V(\boldsymbol{x})$ *around* each point where V is non-differentiable,
- and take the convex hull of all discovered $\nabla V($"around $\boldsymbol{x}$"$)$ as $\partial V(\boldsymbol{x})$.

[2] $\forall x \in \mathbb{R}^d \exists R, C(x) \in \mathbb{R}_+ \forall y \in \mathfrak{B}(x, R) : |V(x) - V(y)| \leq C(x)\|x - y\|$

Set-valued Lie Derivative [Cor08, BC99]: For a locally Lipschitz $V : \mathbb{R}^d \to \mathbb{R}$ and system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, the **set-valued Lie derivative** $\tilde{\mathcal{L}}_f V(\boldsymbol{x})$ of V w.r.t. system $\langle \mathbb{R}^d, f \rangle$ is defined as

$$\tilde{\mathcal{L}}_f V(\boldsymbol{x}) = \{a \in \mathbb{R} \mid \forall \boldsymbol{\xi} \in \partial V(\boldsymbol{x}) : \langle \boldsymbol{\xi}, f(\boldsymbol{x}) \rangle = a\}$$

Set-valued Lie Derivative [Cor08, BC99]: For a locally Lipschitz $V : \mathbb{R}^d \to \mathbb{R}$ and system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, the **set-valued Lie derivative** $\tilde{\mathcal{L}}_f V(\boldsymbol{x})$ of V w.r.t. system $\langle \mathbb{R}^d, f \rangle$ is defined as

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$$\mathcal{L}_f V(\boldsymbol{x}) = \langle \boldsymbol{\xi}, f(\boldsymbol{x}) \rangle, \; \boldsymbol{\xi} = \nabla V(\boldsymbol{x}) \quad \text{– compare}$$

Generalized Invariance Principle $\sim$ [Cor08, BC99]

If the following holds

(i) $V : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz and regular, (was $V \in C^1(D)$)

(ii) $S \subset \mathbb{R}^d$ is compact and invariant for system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, and

(iii) $\max \tilde{\mathcal{L}}_f V(\boldsymbol{x}) \leq 0$ for each $\boldsymbol{x} \in S$, (was $\mathcal{L}_f V(\boldsymbol{x}) \leq 0$)

then all solutions $\boldsymbol{x} : [0, \infty) \to \mathbb{R}^d$ starting in S converge to the largest invariant subset M of

$$S \cap \mathbf{cl}\{x \in \mathbb{R}^d \mid 0 \in \tilde{\mathcal{L}}_f V(\boldsymbol{x})\} \qquad \text{(was } S \cap \{\boldsymbol{x} \in \mathbb{R}^d \mid \mathcal{L}_f V(\boldsymbol{x}) = 0\}\text{)}$$

If M is finite, then the limit of each solution $\boldsymbol{x}(0) \in S$ exists and is an element of M.

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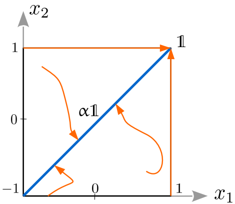
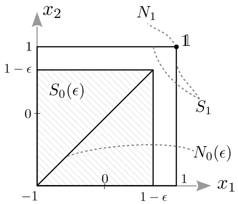
Convergence analysis of the model with stubborn positives

- Model

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= -\boldsymbol{A}(\boldsymbol{x})\boldsymbol{L}\boldsymbol{x}, \text{ -- general model,}\\
\boldsymbol{A}(\boldsymbol{x}) &= (\boldsymbol{I} - \mathbf{diag}(x))/2,\\
\dot{\boldsymbol{x}} &= -1/2(\boldsymbol{I} - \mathbf{diag}(x))\boldsymbol{L}\boldsymbol{x} \text{ -- model with stubborn positives.}
\end{aligned}
$$

- State space partition:





$$
[-1,1]^n = \lim_{\varepsilon\to+0} S_0(\epsilon) \cup S_1,
$$

$$
S_0(\epsilon) = [-1, 1-\epsilon]^n,\ N_0(\epsilon) = \{\alpha\mathbb{1} \mid \alpha \in [-1, 1-\epsilon]\},
$$

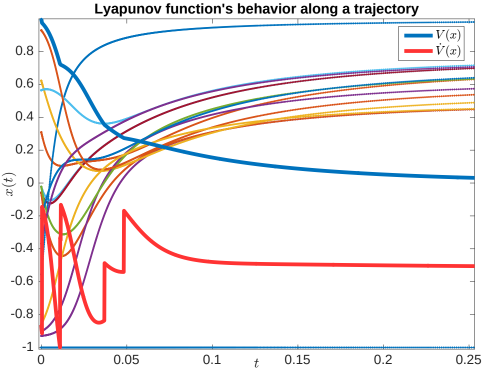
$$
S_1 = \{\boldsymbol{P}^{\mathsf{T}}\boldsymbol{x} \mid \boldsymbol{x} \in \cup_{k=1}^{n}\{1\}^k \times [-1,1)^{n-k}]\},\ N_1 = \mathbb{1}.
$$

Convergence analysis of the model with stubborn positives: $S_0(\epsilon) \rightsquigarrow N_0(\epsilon)$

- Convergence from $S_0(\epsilon) = [-1, 1-\epsilon]^n$ to $N_0(\epsilon) = \{\alpha \mathbb{1} \mid \alpha \in [-1, 1-\epsilon]\}$.

- Convergence from $S_0(\epsilon) = [-1, 1-\epsilon]^n$ to $N_0(\epsilon) = \{\alpha\mathbb{1} \mid \alpha \in [-1, 1-\epsilon]\}$.
- **Lyapunov function candidate** [Mor04, Hen08, Bul16]

$$V_{max-min}(\boldsymbol{x}) = \max(\boldsymbol{x}) - \min(\boldsymbol{x}).$$



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- **Lyapunov function candidate** [Mor04, Hen08, Bul16]

$$V_{max-min}(\boldsymbol{x}) = \max(\boldsymbol{x}) - \min(\boldsymbol{x}).$$

- Computing $\partial V(\boldsymbol{x})$:

$$\partial(c_1 g_1(\boldsymbol{x}) + c_2 g_2(\boldsymbol{x})) = c_1 \partial g_1(\boldsymbol{x}) + c_2 \partial g_2(\boldsymbol{x}) \qquad (*)$$

$$\partial \max\{g_1(\boldsymbol{x}), \dots, g_k(\boldsymbol{x})\} = \textbf{co} \cup \{\partial g_i \mid g_i \text{ attains } \max \text{ at } \boldsymbol{x}\} \qquad (**)$$

$$g_i(\boldsymbol{x}) = \boldsymbol{x}_i, i \in \{1, \dots, n\} \text{ – locally Lipschitz, regular}$$

$$\partial V_{max}(\boldsymbol{x}) = \partial(\max(\boldsymbol{x})) = \boldsymbol{P}^\mathsf{T}[\boldsymbol{\alpha}^\mathsf{T}, \mathbb{0}^\mathsf{T}]^\mathsf{T}, \ \alpha \text{ – “convex”}$$

$$\partial V_{-min}(\boldsymbol{x}) = \partial(-\min(\boldsymbol{x})) = \boldsymbol{P}^\mathsf{T}[-\boldsymbol{\beta}^\mathsf{T}, \mathbb{0}^\mathsf{T}]^\mathsf{T}, \ \beta \text{ – “convex”}$$

$$\partial V_{max-min}(\boldsymbol{x}) = \boldsymbol{P}^\mathsf{T}[\boldsymbol{\alpha}^\mathsf{T}, -\boldsymbol{\beta}^\mathsf{T}, \mathbb{0}^\mathsf{T}]^\mathsf{T}$$

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- Convergence from $S_0(\epsilon) = [-1, 1-\epsilon]^n$ to $N_0(\epsilon) = \{\alpha \mathbb{1} \mid \alpha \in [-1, 1-\epsilon]\}$.
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$$V_{max-min}(\boldsymbol{x}) = \max(\boldsymbol{x}) - \min(\boldsymbol{x}).$$

- Computing $\partial V(\boldsymbol{x})$:

$$\partial V_{max-min}(\boldsymbol{x}) = \boldsymbol{P}^\mathsf{T}[\boldsymbol{\alpha}^\mathsf{T}, -\boldsymbol{\beta}^\mathsf{T}, \mathbb{0}^\mathsf{T}]^\mathsf{T}$$

- Computing $\tilde{\mathcal{L}}_f V_{max-min}(\boldsymbol{x})$:

$$\tilde{\mathcal{L}}_f V_{max-min}(\boldsymbol{x}) = \{a \in \mathbb{R} \mid \forall \boldsymbol{\xi} \in \partial V(\boldsymbol{x}) : \langle \boldsymbol{\xi}, f(\boldsymbol{x}) \rangle = a\}$$

$$\boldsymbol{\xi}^\mathsf{T} f(\boldsymbol{x}) = -\begin{bmatrix} \boldsymbol{\alpha} \\ -\boldsymbol{\beta} \\ \mathbb{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_{max}(\boldsymbol{x}) & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \boldsymbol{A}_{min}(\boldsymbol{x}) & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \boldsymbol{A}_{mid}(\boldsymbol{x}) \end{bmatrix} \times$$

$$\times \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{W}_{11}) & -\boldsymbol{W}_{12} & -\boldsymbol{W}_{13} \\ -\boldsymbol{W}_{21} & (\boldsymbol{I} - \boldsymbol{W}_{22}) & -\boldsymbol{W}_{23} \\ -\boldsymbol{W}_{31} & -\boldsymbol{W}_{32} & (\boldsymbol{I} - \boldsymbol{W}_{33}) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{max} \\ \boldsymbol{x}_{min} \\ \boldsymbol{x}_{mid} \end{bmatrix} =$$

$$= -(\boldsymbol{\alpha}^\mathsf{T} \boldsymbol{A}_{max}(\boldsymbol{x})(\boldsymbol{x}_{max} - [\boldsymbol{W}_{11} \boldsymbol{W}_{12} \boldsymbol{W}_{13}]\boldsymbol{x}) +$$

$$\boldsymbol{\beta}^\mathsf{T} \boldsymbol{A}_{min}(\boldsymbol{x})([\boldsymbol{W}_{21} \boldsymbol{W}_{22} \boldsymbol{W}_{23}]\boldsymbol{x} - \boldsymbol{x}_{min})) \leq 0.$$

- Convergence from $S_0(\epsilon) = [-1, 1-\epsilon]^n$ to $N_0(\epsilon) = \{\alpha\mathbb{1} \mid \alpha \in [-1, 1-\epsilon]\}$.
- **Lyapunov function candidate** [Mor04, Hen08, Bul16]

$$V_{max-min}(\boldsymbol{x}) = \max(\boldsymbol{x}) - \min(\boldsymbol{x}).$$

- Computing $\partial V(\boldsymbol{x})$:

$$\partial V_{max-min}(\boldsymbol{x}) = \boldsymbol{P}^{\mathsf{T}}[\boldsymbol{\alpha}^{\mathsf{T}}, -\boldsymbol{\beta}^{\mathsf{T}}, \mathbb{0}^{\mathsf{T}}]^{\mathsf{T}}$$

- Computing $\tilde{\mathcal{L}}_f V_{max-min}(\boldsymbol{x})$:

$$\max \tilde{\mathcal{L}}_f V_{max-min}(\boldsymbol{x}) \leq 0,$$

$N_0(\epsilon)$ – 0-level set of $\tilde{\mathcal{L}}_f V_{max-min}(\boldsymbol{x})$.

Convergence analysis of the model with stubborn positives: $S_0(\epsilon) \rightsquigarrow N_0(\epsilon)$

- Convergence from $S_0(\epsilon) = [-1, 1-\epsilon]^n$ to $N_0(\epsilon) = \{\alpha\mathbb{1} \mid \alpha \in [-1, 1-\epsilon]\}$.
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- Computing $\partial V(\boldsymbol{x})$:

$$\partial V_{max-min}(\boldsymbol{x}) = \boldsymbol{P}^\mathsf{T}[\boldsymbol{\alpha}^\mathsf{T}, -\boldsymbol{\beta}^\mathsf{T}, \mathbb{0}^\mathsf{T}]^\mathsf{T}$$

- Computing $\tilde{\mathcal{L}}_f V_{max-min}(\boldsymbol{x})$:

$$\max \tilde{\mathcal{L}}_f V_{max-min}(\boldsymbol{x}) \leq 0,$$

$$N_0(\epsilon) \text{ – } 0\text{-level set of } \tilde{\mathcal{L}}_f V_{max-min}(\boldsymbol{x}).$$

- Invariance of compact $S_0(\epsilon)$: comes at no cost from $\max \tilde{\mathcal{L}}_f V_{max}(\boldsymbol{x}) \leq 0$.

Convergence analysis of the model with stubborn positives: $S_0(\epsilon) \rightsquigarrow N_0(\epsilon)$

- Convergence from $S_0(\epsilon) = [-1, 1-\epsilon]^n$ to $N_0(\epsilon) = \{\alpha\mathbb{1} \mid \alpha \in [-1, 1-\epsilon]\}$.
- **Lyapunov function candidate** [Mor04, Hen08, Bul16]

$$V_{max-min}(\boldsymbol{x}) = \max(\boldsymbol{x}) - \min(\boldsymbol{x}).$$

- Computing $\partial V(\boldsymbol{x})$:

$$\partial V_{max-min}(\boldsymbol{x}) = \boldsymbol{P}^\top [\boldsymbol{\alpha}^\top, -\boldsymbol{\beta}^\top, \mathbb{0}^\top]^\top$$

- Computing $\tilde{\mathcal{L}}_f V_{max-min}(\boldsymbol{x})$:

$$\max \tilde{\mathcal{L}}_f V_{max-min}(\boldsymbol{x}) \leq 0,$$

$N_0(\epsilon)$ – 0-level set of $\tilde{\mathcal{L}}_f V_{max-min}(\boldsymbol{x})$.

- Invariance of compact $S_0(\epsilon)$: comes at no cost from $\max \tilde{\mathcal{L}}_f V_{max}(\boldsymbol{x}) \leq 0$.
- From Invariance Principle, all solutions starting in $S_0(\epsilon)$ converge asymptotically to $N_0(\epsilon)$.

Convergence analysis for other models

- For other part(s) of the state space and other models, use same techniques.
- Lyapunov functions based on a combination of V_{max}, V_{-min} usually work.

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▶ Summary

Summary

- Polar opinion dynamics can be modeled with non-linear dynamical system whose behavior can be formally analyzed.
- Generally, non-smooth Lyapunov functions can be applied to the stability analysis of continuous non-linear systems over directed networks.
- Our convergence proofs did not rely on a particular shape of $\boldsymbol{A}(\boldsymbol{x})$ (only when its diagonal elements are zero / non-zero) and, thus, can be carried over to other models of similar form.
- Similar techniques can be used to design Lyapunov functions out of convex components.

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$\sim$ Thanks $\sim$

Questions?

[Backup] Convergence: general model $\dot{\boldsymbol{x}} = -\boldsymbol{A}(\boldsymbol{x})\boldsymbol{L}\boldsymbol{x}$

Let $S \subseteq [-1,1]^n$ be a non-empty compact forward-invariant set, and

$$N = S \ \cap \ \{\alpha\mathbb{1} \mid \alpha \in [-1,1]\}$$

be its non-empty subset of consensus states. Further, assume that in S, the agents' susceptibility functions $\boldsymbol{A}_{ii}(\boldsymbol{x})$ agree upon their zeros in that

$$\forall \boldsymbol{x} \in S \ \forall i,j \in \{1,\dots,n\} : \boldsymbol{A}_{ii}(\boldsymbol{x}) = \boldsymbol{A}_{jj}(\boldsymbol{x}) = 0 \to x_i = x_j.$$

Then, all trajectories $\boldsymbol{x}(t)$ of $\dot{\boldsymbol{x}} = -\boldsymbol{A}(\boldsymbol{x})\boldsymbol{L}\boldsymbol{x}$ starting in S converge to N as $t \to \infty$.